Coexistence of hyperbolic and nonhyperbolic chaotic scattering

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(Received 16 February 1993)

Chaotic scattering at different projectile incident energies is studied for a model which involves a two-body van der Waals–type interaction. At higher energies one finds characteristics typical for hyperbolic chaotic scattering. For sufficiently low energies hyperbolic and nonhyperbolic chaotic scattering are found to coexist at the same energy. The mechanism of this coexistence is discussed in terms of the Lyapunov exponent and the fractal dimension. Arguments are put forward for an increase in the fractal dimension of the set of singularities leading to nonhyperbolic chaotic scattering.

PACS number(s): 05.45.+b, 03.20.+i, 03.65.Nk

A most natural and important source of information in physics is scattering experiments. This fact points to the importance of a detailed understanding of the mechanism of scattering at the classical level, which, with the help of the semiclassical arguments, can shed more light on generic properties of the quantum counterparts. It is by now well recognized [1] that the universal feature of scattering in classical mechanics is irregularity. Since in scattering a trajectory stays in the interaction region for finite time only, one refers to the resulting chaotic motion as "transient chaos." It manifests itself in an extremely strong sensitivity of the scattering angle θ or the delay time on impact parameter [2]. An interesting aspect of such processes is the appearance of the fractal set of singularities [3]. The underlying dynamics can be classified as either hyperbolic or nonhyperbolic. In the hyperbolic chaotic scattering there are no tori in the scattering region and all periodic orbits are unstable. The typical delay-time probability behaves exponentially in this case [4]. On the quantum level, this corresponds [5] to Ericson fluctuations [6]. The distribution of resonance poles in the complex energy plane is conjectured [7] to generically reveal the cubic pole repulsion. Nonhyperbolic chaotic scattering [8], on the other hand, is connected to the presence of Kolmogorov-Arnold-Moser (KAM) surfaces and, as current evidence indicates, is characterized by power-law decay [9]. On the quantum level, this leads [10] to a cusp shape of the autocorrelation function typical for isolated resonances. The nature of nonhyperbolic chaotic scattering is, however, not so well understood as in the hyperbolic case.

In view of both universal and practical aspects of the above problems, in this paper we address the question of classical chaotic scattering using a model which involves the basic ingredients of realistic physical systems and which simultaneously allows us to perform a fully quantitative and extensive numerical analysis. Many effective interactions in physical systems are well described by potentials which are of the van der Waals type, i.e., they include weak long-range repulsion, intermediate-range attraction, and short-range strong repulsion. Such a form of the potential is therefore adopted here to generate the two-body interaction. Its precise shape corresponds to the α-α interaction and is specified in Ref. [11]. Our system is composed of four bodies. Three of them are frozen in the plane at the corners of an equilateral triangle and form the target. The corresponding equipotential lines are shown in Fig. 1. The fourth particle, restricted to the same plane, constitutes the projectile coming from the right with initial momentum parallel to the x axis. The mass of the projectile equals the mass of the α particle (nucleus of ^{4}He).

At higher scattering energy where the projectile is affected by essentially only the repulsive cores, generated by the three centers, one obtains an already familiar picture. In certain regions of the impact parameter the delay time displays a very irregular behavior and the singularities form a fractal set, as is illustrated in the left part of Fig. 2 for an energy of $E = 12$ MeV. Here the fractal dimension D is uniquely defined and application of the

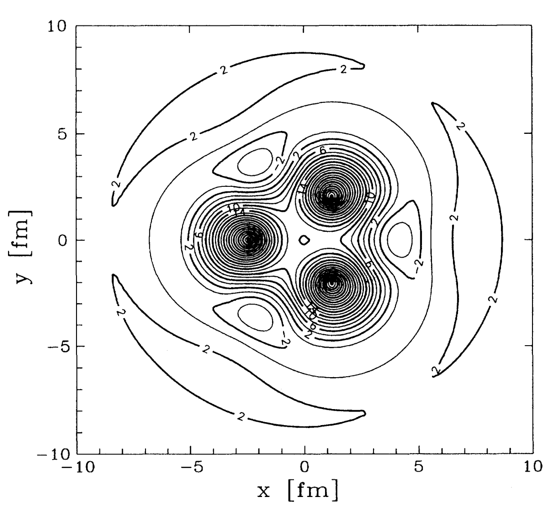

FIG. 1. Contour plot of the equipotential lines of the target, shown at intervals of 2 MeV. The centers are located at $(-\sqrt{3}a/3, 0)$, $(\sqrt{3}a/6, a/2)$ and $(\sqrt{3}a/6, -a/2)$ respectively, where $a = 4.1$.

uncertainty exponent technique [12, 13] gives $D = 0.64$. A pair of trajectories is counted as uncertain if they differ in θ by more than $\pi/2$. The resulting "uncertainty dimension" D [13] equals the capacity dimension D_0 [4].

Following the concepts of the transport theory [14], by uniform random sampling of the whole interval of impact parameters one determines the number $N(t)$ of trajectories remaining in the interaction region up to time t. At E=12 MeV we observe that the asymptotic dependence (dotted line) is exponential. The solid line is calculated independently, according to the prescription [15]

$$N(t) = N_0 \exp[-\lambda(1-D)t], \qquad (1)$$

where λ denotes the Lyapunov exponent and N_0 the total number of trajectories. Justification for such a prescription comes from the following consideration. The phenomenon under consideration can be visualized as involving two ingredients. One is the geometry of the fractal set connected to the form of the potential. Consistent with the structure of such a set it is natural to expect that the number N of trajectories remaining in the interaction region after the ith iteration is proportional to the total length of unremoved intervals. The removed intervals after the ith iteration correspond to regular regions in Fig. 2 after appropriate magnification, while those unremoved measure the dimension of singularities as $i \to \infty$. Thus $N \sim m(i)\epsilon$ where m is the number of remaining elementary intervals and ϵ is the length of one such interval. Since $m = e^{\ln m} = \epsilon^{\ln m/\ln \epsilon} = \epsilon^{-D}$, this yields $N \sim \epsilon^{1-D}$. In this case D represents the capacity dimension D_0 [4]. Possible nonuniformity in the fractal structure results in replacing $\ln m$ by $\sum_{j=1}^{m} P_j \ln P_j$, where $\{P_j\}$ describes partitioning of the trajectories over the unremoved intervals. Thus, in general, D in formula (1) is the information dimension D_1 ($D_1 \leq D_0$) [4].

The second ingredient, the dynamics, describes the speed of the above removal and fixes the time scale between consecutive iterative steps. The way time intervenes can be inferred from the relation $\epsilon(i+1) = \epsilon(i)\exp(-\lambda\Delta t)$, which holds asymptotically and where Δt expresses how long it takes to evolve from the ith to the $(i+1)$th removal. Combining those two ingredients one obtains Eq. (1).

To verify Eq. (1), in addition to the numerical value of D, one also needs a precise value of the λ. A standard procedure [16] is to evolve an initial unit vector δ in tangent space along a selected trajectory and to calculate $\lambda(t) = \ln|\delta|/t$. Then $\lambda = \lim_{t\to\infty}\lambda(t)$ which requires that the evolution has to be executed long enough, until convergence is reached. For a scattering problem, characterized by exponential decay, there exists no practical means to get a trajectory that remains sufficiently long in the interaction region. In an exponentially unstable case (the system loses memory) one expects, however, a significant acceleration of the convergence by taking $\lambda(t) = \frac{1}{n}\Sigma_{i=1}^{n}\lambda_i(t)$, where n denotes a certain number of randomly chosen trajectories. We have tested the equivalence of both procedures for the logistic map with positive result. For the scattering problem considered here, averaging over trajectories remaining in the interaction region up to time t and analyzing such an average as a function of t allows us to evaluate λ very precisely (see inset to Fig. 2). One obtains $\lambda = 0.0863$ (in units of the inverse of 10^{-23} s). Inserting the above values of D and λ into (1) gives the straight line in Fig. 2, which perfectly agrees with the empirical result and provides a consistency test.

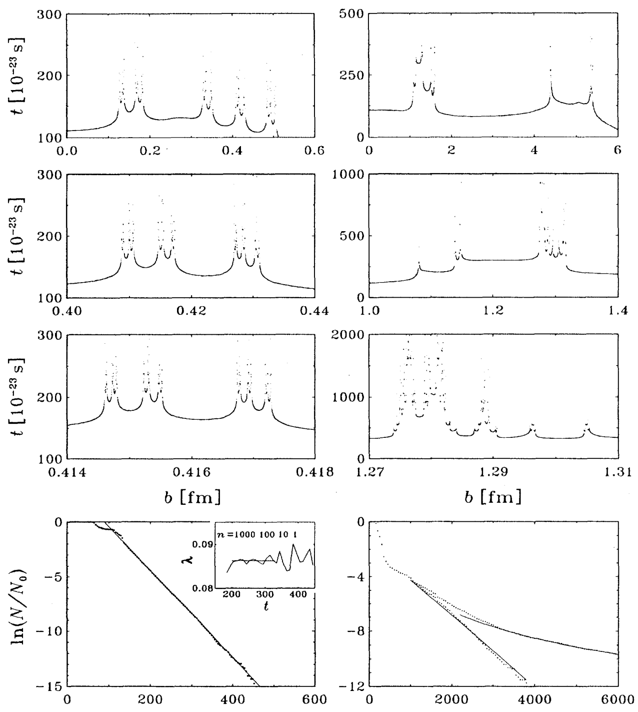

FIG. 2. Upper part: Delay time versus impact parameter b for scattering off the target as shown in Fig. 1 for $E = 12$ MeV (l.h.s) and $E = 3.5$ MeV (r.h.s.). Lower part: the corresponding $N(t)$ for ensembles of 5×10^6 trajectories sampling uniformly impact parameters from the interval (0.0, 0.6) covering all irregularities for $E = 12$ MeV and from the interval (1.0, 1.6) for $E = 3.5$ MeV. The dotted lines refer to the numerical experiment, straight-solid lines are obtained using Eq. (1) and the solid line is a fit reflecting asymptotic power behavior (index -2.84) of $N(t)$ at $E = 3.5$ MeV. The inset illustrates the way in which λ is evaluated by ensemble average. The integer n refers to the number of trajectories used, surviving up to time t.

At lower incident energies ($E \leq 8$ MeV) the situation becomes more complicated. As shown on the right-hand part of Fig. 2, for $E = 3.5$ MeV, the delay time versus impact parameter still displays a kind of self-similar structure, but now the survival probability shows a trace of an exponential behavior only for short times, up to 2.5×10^{-21} s. For longer times the decay is described by the power law, $N(t) = N_0 t^{-z}$, with $z = 2.84$. Thus, we face the coexistence of two different mechanisms of chaotic scattering, hyperbolic and nonhyperbolic, at the same energy.

Several questions arise. What initial conditions preferably lead to long trajectories? What is a nature of the corresponding set of singularities? And finally, which in-

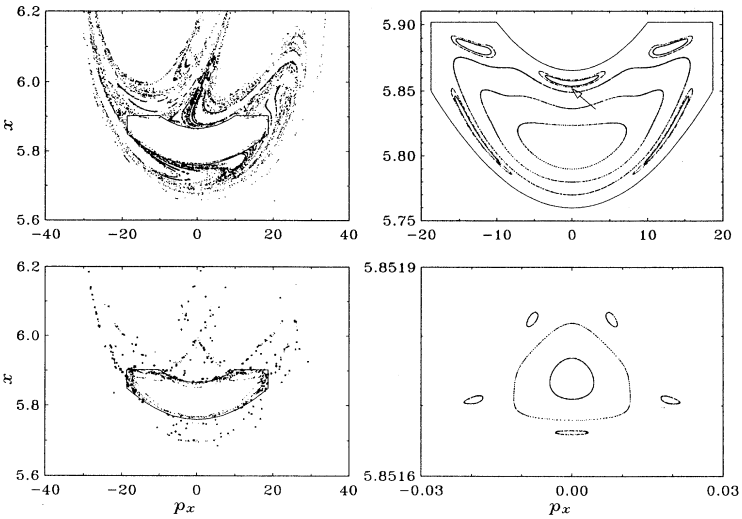

FIG. 3. Surface of section through the interaction region in x and p_x at $y = 0$ with separation into shorter (10^{-21} s< $t < 2.5\times10^{-21}$ s) (upper left) and longer ($t > 3.5 \times 10^{-21}$s) (lower left) trajectories. The solid line defines an envelope (see text). The symbols + and × denote the points at which the trajectories enter and leave the section, respectively. The right hand part illustrates the structure of the envelope studied using trajectories initialized inside. The lower part shows a magnification of the little dot indicated by the arrow.

gredient, the fractal dimension D or the Lyapunov exponent λ alters Eq. (1), to account for the observed survival probability?

Trying to answer the first question we find that there exists no obvious distinction in impact parameters which would lead to either short or long trajectories. All of them are distributed in a similar way. More instructive is analysis of the structure of the phase space in the interaction region at the surface of section. One such section in x and p_x at $y = 0$ is shown in Fig. 3 with the separation into shorter (10^{-21} s< $t < 2.5 \times 10^{-21}$ s, upper left corner) and longer ($t > 3.5 \times 10^{-21}$ s, lower left part) trajectories. Such a separation defines an envelope (indicated approximately by the solid line); no short trajectory enters inside while all longer ones, from the power tail, do. The content of the envelope cannot, however, be studied systematically with the help of those trajectories alone. All of them stay in its outer part. The toroidal structures displayed on the right-hand part of Fig. 3 are traced by the trajectories initialized inside the envelope. These structures reveal self-similarity also.

Further evidence for the importance of the region inside the envelope comes from the following exercise. Eliminating the influence of this region by no longer counting the scattering trajectories entering it, or equivalently, by assigning to them the time they enter the envelope as a decay time, again results in exponential decay. Moreover, as documented in Fig. 2, this feature is also satisfactorily reproduced by Eq. (1), using the corresponding D which here equals 0.66 and $\lambda = 0.0077$ (calculated by ensemble average). Thus, the region of phase space which causes an extra delay is populated with trajectories originating from exponential decay of the ordinary transient. The two-step character of the observed process essentially explains why initial conditions leading to long trajectories (decaying according to a power law), have the same distribution as all initial conditions corresponding to irregular scattering.

Lau, Finn, and Ott suggest [13] that the fractal dimension associated with the set of singularities leading to nonhyperbolic chaotic scattering approaches 1. The required set is modeled by a Cantor-like construction, but by removing at the ith stage a fraction $\eta_i = \alpha/(i + c)$ (α and c are constants). Such a fraction represents the ratio of trajectories escaping at the ith iteration. For constant λ, this is equivalent to the inverse proportionality to time of such a ratio which converts Eq. (1) into a time dependence which follows a power law. This becomes clear by rewriting the corresponding equations in differential form. In fact, Fig. 2 seems to support the existence of the above kind of fractal set at $E = 3.5$ MeV. Qualitatively, the density of singularities in successive blowups, involving the longer trajectories, is larger, while for $E = 12$ MeV it remains unchanged. However, another possibility to effectively get the same result is to have the same fraction η_i at each stage but appropriately slowing down the speed of removal (λ inversely proportional to time). In fact, the λ calculated along the few longest scattering trajectories identified at $E = 3.5$ MeV converges to an even lower (as compared to 0.0077) value of about 0.0025. However, no lower value than this has been found and $\lambda = 0.0021$ seems to be a lower bound on the Lyapunov exponent of scattering trajectories. This is the λ for the trajectory tracing the outermost closed curve shown in the upper right corner of Fig. 3. When going inside, the Lyapunov exponents decrease almost linearly

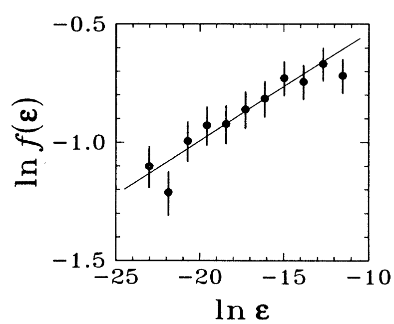

FIG. 4. An illustration of how the fraction of uncertain pairs scales as a function of the relative distance ϵ for trajectories initialized in the interval $x \in (-5.851\,65, -5.850\,05)$, at $y = 0$.

with x and approach zero at the innermost curve shown. But all scattering trajectories considered, although they clearly determine the power-law decay, do not enter so deeply. This provides indirect evidence for an increase of the fractal dimension.

Reliable explicit evaluation of D based on long scattering trajectories alone would be extremely difficult because the probability of an appearance of such trajectories is suppressed by many orders of magnitude. However, sufficiently many long trajectories can be generated by initializing them inside the envelope. Then, a similar analysis as in Ref. [13], i.e., gradually improving the resolution, results in a systematic increase of the uncertainty dimension. Specifying initial conditions for x in the interval $(-5.851\,65, -5.850\,05)$ (at y=0), which is on the borderline set by numerical precision, gives $D \approx 0.95$ (see Fig. 4). This tendency, together with a finite lower limit identified for λ and the observed asymptotic behavior of the survival probability, leads us to the conclusion that the fractal dimensions, both D_0 and D_1, approach 1 (but the Lebesgue measure remains zero), which confirms and extends the suggestion of Lau, Finn, and Ott [13]. The comparatively large value of the power index in the present model indicates, however, that such an approach is very slow and the precise limit of unity may remain unreachable by numerical means.

We thank Jochen Wambach for reading the manuscript and for helpful suggestions. This work was supported by KBN Grant No. 2 P302 157 04.

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